THE THEORY OF THE STATIONARY MODES OF CYLINDRICAL AND SPHERICAL DIODES

V. A. Syrovoi

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The construction of a theory of cylindrical and spherical diodes during emission, limited to the space charge, was begun in [1–3]. The cycle of papers which followed upon these investigated the various modes which could be realized in these devices. The solution was obtained either by expansion in series, in which some function of the radius served as the expansion parameter, or by numerical integration of the beam equation; for the spherical diode the solution was given by the Airy functions. Recurrence relations are given in [4, 5] for the expansion coefficients for any parameter in which the expansion is carried out, and for arbitrary geometry. However, the approach which was used in the above-mentioned papers gave rise to well-known difficulties in determining the time of flight of the particles. These difficulties were removed by the introduction of a time formalism, first suggested in [6] and used in [7, 8] to study cylindrical and spherical diodes in the mode of total space charge. An analogous problem is solved below under arbitrary emission conditions, and the recurrence relations for the series coefficients are given. The tensor form of the beam equation leaves us free to choose the parameter to be used in the expansion for the time of flight.

§1. The radial motion of charges having the same value and sign of specific charge η in the space between the coaxial cylinders or between the concentric spheres, is described by a system of equations which in tensor notation have the form

$$v^1 \frac{dv^1}{dx} + \frac{1}{2}\frac{d \ln g_{11}}{dx}(v^1)^2 = g^{11}\frac{d\varphi}{dx},$$
$$\sqrt{g}\,\rho v^1 = J g_{22}(0)\, g_{33}(0), \qquad \frac{1}{\sqrt{g}}\frac{d}{dx}\left(\sqrt{g}\, g^{11}\frac{d\varphi}{dx}\right) = \rho. \quad (1.1)$$

Here v^1 is the contravariant component of the velocity, φ is the scalar potential, ρ is the space charge density, and J = const is the density of the emission current. The symbol g denotes the radial part of the determinant of the metric tensor g_{ik}, and x is some function of the radius such that x = 0 at the emitter. Equation (1.1) is written in terms of the dimensionless variables r^0, v^0, φ^0, ρ^0 (r^0 and v^0 are the moduli of the radius vector and the velocity vector):

$$r = ar^\circ, \qquad v = Uv^\circ, \qquad \varphi = -\frac{U^2}{\eta}\varphi^\circ, \qquad \rho = \frac{U^2}{4\pi\eta a^2}\rho^\circ,$$

in which the symbol of a dimensionless quantity was omitted; a and U are constants, having the dimensions of length and velocity, respectively It is convenient to choose for a the radius of the emitter. The introduction of the time formalism

$$\frac{d}{dx} = \frac{1}{v^1}\frac{d}{dt}$$

permits us to reduce system (1.1) to the single equation

$$\sqrt{g}\left(x^{\cdot\cdot} + \frac{1}{2}\frac{d \ln g_{11}}{dx}x^{\cdot 2}\right) = (Jt + \varepsilon) b_0^{1/2} c_0^{1/2},$$
$$b_0 = g_{22}(0), \; c_0 = g_{33}(0). \quad (1.2)$$

Here $\varepsilon = (g^{11})^{1/2} d\varphi/dx$ = const is the electric field at the emitter. We shall write out below the solutions of Eq. (1.2) for cylindrical and spherical diodes (where x is taken to be simply the radius) under arbitrary emission conditions.

For a cylindrical diode x = R − 1, $g_{11} = 1$, $(g)^{1/2} = R$, $b_0 = c_0 = 1$, and Eq. (1.2) takes the form

$$RR^{\cdot\cdot} = Jt + \varepsilon. \quad (1.3)$$

We shall write the solution of Eq. (1.3) in series form:

$$R = \alpha_k t^k \quad (k = 0, 1, \ldots), \quad (1.4)$$

the coefficients of which must obey the recurrence relation

$$\sum_{l=0}^{s} (l+1)(l+2)\alpha_{l+2}\alpha_{s-l} - J\delta_{1s} - \varepsilon\delta_{0s} = 0$$
$$(s = 0, 1, \ldots). \quad (1.5)$$

Here $\delta_{as} = 0$ when $a \neq s$ and $\delta_{as} = 1$ when $a = s$. Noting that R = 1 and $R^{\cdot} = v_0$ at t = 0, we obtain $\alpha_0 = 1$, $\alpha_1 = v_0$, where v_0 is the particle velocity at the emitter. Let us write out a few terms of the expansion (1.4):

$$R = 1 + v_0 t + {}^1/_2\, \varepsilon t^2 + {}^1/_6\, (J - \varepsilon v_0)\, t^3 -$$
$$- {}^1/_{12}\, [{}^1/_2\, \varepsilon^2 + v_0\, (J - \varepsilon v_0)]\, t^4 -$$
$$- {}^1/_{20}\, [{}^2/_3\, \varepsilon J - {}^7/_6\, \varepsilon^2 v_0 - v^2_0\, (J - \varepsilon v_0)]\, t^5 -$$
$$- {}^1/_{30}\, [-{}^7/_{24}\, \varepsilon^3 + {}^{23}/_{12}\, \varepsilon^2 v^2_0 -$$
$$- {}^{19}/_{12}\, \varepsilon J v_0 + {}^1/_6\, J^2 + v_0^3\, (J - \varepsilon v_0)]\, t^6 + \ldots. \quad (1.6)$$

For a spherical diode x = r − 1, $g_{11} = 1$, $(g)^{1/2} = r^2$, $b_0 = c_0 = 1$, and r(t) satisfies the equation

$$r^2 r^{\cdot\cdot} = Jt + \varepsilon. \quad (1.7)$$

The solution of Eq. (1.7) is given by a series

$$r = \beta_k t^k \quad (k = 0, 1, \ldots) \quad (1.8)$$

with coefficients determined by recurrence relations of the form

$$\sum_{l=0}^{s} (l+1)(l+2)\beta_{l+2}\gamma_{s-l} - J\delta_{1s} - \varepsilon\delta_{0s} = 0 \quad (s = 0, 1, \ldots),$$
$$\gamma_{2k} = \beta_k^2 + 2\sum_{l=1}^{k} \beta_{l-1}\beta_{2k-l+1}, \qquad \gamma_{2k+1} = 2\sum_{l=0}^{k} \beta_l \beta_{2k-l+1}. \quad (1.9)$$

Using Eq. (1.9), we obtain

$$r = 1 + v_0 t + {}^1/_2\, \varepsilon t^2 + {}^1/_6\, (J - 2\varepsilon v_0)\, t^3 -$$
$$- {}^1/_{12}\, [\varepsilon^2 + v_0\, (2J - 3\varepsilon v_0)]\, t^4 -$$
$$- {}^1/_{20}[{}^4/_3\, \varepsilon J - {}^{11}/_3\, \varepsilon^2 v_0 - v_0^2\, (3J - 4\varepsilon v_0)]\, t^5 -$$
$$- {}^1/_{30}\, [-{}^{11}/_{12}\, \varepsilon^3 + {}^{17}/_2\, \varepsilon^2 v_0^2 - 5\varepsilon J v_0 +$$
$$+ {}^1/_3\, J^2 + v_0^3\, (4J - 5\varepsilon v_0)]\, t^6 + \ldots. \quad (1.10)$$

It is interesting to note that Eqs. (1.6) and (1.10) are correct under arbitrary conditions at the emitter surface. The special cases of these formulas are the expressions given in [7, 8]. Note also that Eqs. (1.6) and (1.10) can be used to determine the radius corresponding to a given time of flight t. We shall consider below the problem of finding the explicit dependence of t = t(x).

§2. The inversion of Eq. (1.2) with the help of the relation $x^{\cdot\cdot} = -t''/t'^3$ leads to the following result:

$$\sqrt{g}\left(t'' - \frac{1}{2}\frac{d \ln g_{11}}{dx}t'\right) + b_0^{1/2} c_0^{1/2} (Jt + \varepsilon)\, t'^3 = 0. \quad (2.1)$$

There are now three cases which must be investigated: emission in the ρ-mode ($\varepsilon = 0, v_0 = 0$), emission in the T-mode ($\varepsilon \neq 0, v_0 = 0$), and emission with a nonzero initial velocity ($v_0 \neq 0$).

Since we propose to find a solution of Eq. (2.1) in the form of a series in $x^{\varkappa}$, let us similarly represent the functions $(g)^{1/2}$ and $(g)^{1/2} \times$ $\times$ d ln g_{11}/dx, which carry information about the system of coordinates being used for the analysis:

$$\sqrt{g} = G_k x^k, \quad \sqrt{g}\, d \ln g_{11} / dx = A_k x^k,$$
$$g_{11} = a_k x^k \quad (k = 0, 1, \ldots). \tag{2.2}$$

For emission in the ρ-mode, we seek a solution of Eq. (2.1) in the series form

$$t = x^{1/3} (\tau_k x^k) \quad (k = 0, 1, \ldots). \tag{2.3}$$

For the determination of the coefficients τ_k we then obtain

$$\sum_{k=0}^{l} \left[\left(k + \frac{1}{3}\right)\left(k - \frac{2}{3}\right) \tau_k G_{l-k} + \frac{1}{2} b_0^{1/2} c_0^{1/2} J \left(k + \frac{2}{3}\right) \beta_k \gamma_{l-k} \right] - \frac{1}{2} \sum_{k=0}^{l-1} \left(k + \frac{1}{3}\right) \tau_k A_{l-k-1} = 0 \quad (l = 0, 1, \ldots),$$
$$\beta_{2s} = \tau_s^2 + 2 \sum_{l=1}^{s} \tau_{l-1} \tau_{2s-l+1}, \quad \beta_{2s+1} = 2 \sum_{l=0}^{s} \tau_l \tau_{2s-l+1},$$
$$\gamma_{2s} = \left[\left(s + \frac{1}{3}\right) \tau_s\right]^2 + 2 \sum_{l=1}^{s} \left(l - \frac{2}{3}\right) \times \left(2s - l + \frac{4}{3}\right) \tau_{l-1} \tau_{2s-l+1},$$
$$\gamma_{2s+1} = 2 \sum_{l=0}^{s} \left(l + \frac{1}{3}\right)\left(2s - l + \frac{4}{3}\right) \tau_l \tau_{2s-l+1}. \tag{2.4}$$

We note that all coefficients with negative indices, as well as the sum from a to b when $a > b$, are equal to zero by definition. Using Eq. (2.4), we obtain for the time of flight

$$t = \left(\frac{6}{J}\right)^{1/3} s^{1/3} \left[1 + \left(\frac{1}{12} \frac{a_1}{a_0^{3/2}} - \frac{1}{15} T\right) s + \left(\frac{1}{18} \frac{a_2}{a_0^2} - \frac{1}{48} \frac{a_1^2}{a_0^3} - \frac{1}{45} \frac{a_1}{a_0^{3/2}} T + \frac{1}{360} T^2 - \frac{1}{72} T_s'\right) s^2 + \ldots\right], \quad s = a_0^{1/2} x. \tag{2.5}$$

Here T is the total curvature of the emitting surface, equal to the sum of the principal curvatures, and T_s' is its radial derivative at $x = 0$ (for the cylinder, $T = -1/R$, and for the sphere, $T = -2/R$).

For emission in the T-mode we shall construct an expansion in half-integral powers of x:

$$t = \tau_k x^{1/2 k} \quad (k = 1, 2, \ldots). \tag{2.6}$$

The coefficients in Eq. (2.6) satisfy the recurrence relations

$$\sum_{k=1}^{l} k \left(\frac{k}{2} - 1\right) \tau_k G_{1/2(l-k)} - \frac{1}{2} \sum_{k=1}^{l-2} k \tau_k A_{1/2(l-k-2)} + \left[\frac{1}{2} J \sum_{k=2}^{l} k \beta_k \gamma_{l-k+2} + \varepsilon \sum_{k=2}^{l+1} (l - k + 2) \gamma_k \tau_{l-k+2}\right] b_0^{1/2} c_0^{1/2} = 0,$$
$$(l = 1, 2, \ldots),$$
$$\beta_{2s} = \tau_s^2 + 2 \sum_{l=1}^{s-1} \tau_l \tau_{2s-l}, \quad \beta_{2s+1} = 2 \sum_{l=1}^{s} \tau_l \tau_{2s-l+1},$$
$$\gamma_{2s} = \left(\frac{1}{2} s \tau_s\right)^2 + \frac{1}{2} \sum_{l=1}^{s-1} l (2s - l) \tau_l \tau_{2s-l},$$
$$\gamma_{2s+1} = \frac{1}{2} \sum_{l=1}^{s} l (2s - l + 1) \tau_l \tau_{2s-l+1}. \tag{2.7}$$

Let us write out the first few coefficients of the expansion, as determined from these formulas:

$$t = \left(\frac{2}{\varepsilon}\right)^{1/2} s^{1/2} - \frac{1}{3} \frac{J}{\varepsilon^2} s + \left(\frac{2}{\varepsilon}\right)^{1/2} \left(\frac{1}{8} \frac{a_1}{a_0^{3/2}} - \frac{1}{12} T + \frac{5}{12} \frac{J^2}{\varepsilon^3}\right) s^{3/2} + \left(-\frac{1}{12} \frac{a_1}{a_0^{3/2}} \frac{J}{\varepsilon^2} + \frac{1}{30} \frac{J}{\varepsilon^2} T - \frac{7}{27} \frac{J^3}{\varepsilon^5}\right) s^2 + \ldots \tag{2.8}$$

It remains only to examine the case of nonzero emission velocity, when the solution of Eq. (2.1) is written in the form

$$t = \tau_k x^k \quad (k = 0, 1, \ldots). \tag{2.9}$$

For the coefficients τ_k we obtain

$$\sum_{k=2}^{l+2} k (k - 1) \tau_k G_{l-k+2} - \frac{1}{2} \sum_{k=1}^{l+1} k \tau_k A_{l-k+1} + \left[\frac{1}{2} J \sum_{k=2}^{l+1} k \beta_k \gamma_{l-k+3} + \varepsilon \sum_{k=2}^{l+2} (l - k + 3) \gamma_k \tau_{l-k+3}\right] b_0^{1/2} c_0^{1/2} = 0 \quad (l = 0, 1, \ldots),$$
$$\beta_{2s} = \tau_s^2 + 2 \sum_{l=1}^{s-1} \tau_l \tau_{2s-l}, \quad \beta_{2s+1} = 2 \sum_{l=1}^{s} \tau_l \tau_{2s-l+1},$$
$$\gamma_{2s} = (s \tau_s)^2 + 2 \sum_{l=1}^{s-1} l (2s - l) \tau_l \tau_{2s-l},$$
$$\gamma_{2s+1} = 2 \sum_{l=1}^{s} l (2s - l + 1) \tau_l \tau_{2s-l+1}. \tag{2.10}$$

Using Eq. (2.10), we find that

$$t = \frac{1}{v_0} s + \left(\frac{1}{4 v_0} \frac{a_1}{a_0^{3/2}} - \frac{1}{2} \frac{\varepsilon}{v_0^3}\right) s^2 + \left(\frac{1}{6 v_0} \frac{a^2}{a_0^2} - \frac{1}{24 v_0} \frac{a_1^2}{a_0^3} - \frac{1}{4} \frac{\varepsilon}{v_0^3} \frac{a_1}{a_0^{3/2}} - \frac{1}{6} \frac{\varepsilon}{v_0^3} T - \frac{1}{6} \frac{J}{v_0^4} + \frac{1}{2} \frac{\varepsilon^2}{v_0^5}\right) s^3 + \ldots \tag{2.11}$$

For the function x in Eqs. (2.5), (2.8), and (2.11) we can use R − − 1, ln R, 1 − 1/R, and so on.

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